## 10.5 Function of Continuous Random Variables: SISO

Reconsider the derived random variable $Y = g(X)$.

Ex $Y = X^2$ $\quad g(\cdot) = (\cdot)^2$

$Y = 4|X-1.5|$ $\quad g(\cdot) = 4|\cdot - 1.5|$

$$X \longrightarrow \boxed{g(\cdot)} \longrightarrow Y = g(X)$$

Recall that we can find $\mathbb{E}Y$ easily by (22):

$$\mathbb{E}Y = \mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f_X(x) dx.$$

$$\mathbb{E}[Y] = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

However, there are cases when we have to evaluate probability directly involving the random variable $Y$ or find $f_Y(y)$ directly.

Recall that for discrete random variables, it is easy to find $p_Y(y)$ by adding all $p_X(x)$ over all $x$ such that $g(x) = y$:

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x). \tag{23}$$

For continuous random variables, it turns out that we can't[47] simply integrate or add the pdf of $X$ to get the pdf of $Y$.

Before doing this, check that $Y$ is cont. by checking that the solutions for $y = g(x)$ is always (at most) countable at every $y$ value.

**10.76.** For $Y = g(X)$, if you want to find $f_Y(y)$, the following ***two-step procedure*** will always work and is easy to remember:

(a) Find the cdf $F_Y(y) = P[Y \le y]$. $= P[g(X) \le y]$

(b) Compute the pdf from the cdf by "finding the derivative" $f_Y(y) = \frac{d}{dy} F_Y(y)$ (as described in 10.22).

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$$

$Y = aX + b$

**Example 10.77.** Suppose $X \sim \mathcal{E}(\lambda)$. Let $Y = 5X$. Find $f_Y(y)$.

Method 1

Step ①: $F_Y(y) = P[Y \le y] = P[5X \le y] = P\left[X \le \frac{y}{5}\right] = F_X\left(\frac{y}{5}\right)$

by defn. of being a cdf

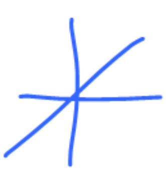
$Y = 5X$

$F_X(x) = P[X \le x]$

chain rule

Step ②: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X\left(\frac{y}{5}\right)\right) = \frac{1}{5} f_X\left(\frac{y}{5}\right) = \frac{1}{5} \times \begin{cases} \lambda e^{-\lambda \frac{y}{5}}, & \frac{y}{5} > 0, \\ 0, & \text{otherwise} \end{cases}$

$$= \begin{cases} \frac{\lambda}{5} e^{-\frac{\lambda}{5} y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$



[47] When you applied Equation (23) to continuous random variables, what you would get is $0 = 0$, which is true but not interesting nor useful.

Observe that $Y \sim \mathcal{E}\left(\frac{\lambda}{5}\right)$

Method 2

$Y = 5X \Rightarrow a = 5, b = 0$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{5} f_X\left(\frac{y}{5}\right)$

(Affine)

**10.78.** ***Linear Transformation***: Suppose $Y = aX + b$. Then, the cdf of $Y$ is given by

step ①

$$F_Y(y) = P[Y \le y] = P[aX + b \le y] = \begin{cases} P\left[X \le \frac{y-b}{a}\right], & a > 0, \\ P\left[X \ge \frac{y-b}{a}\right], & a < 0. \end{cases}$$

Now, by definition, we know that

$$P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right),$$

and

$$P\left[X \ge \frac{y-b}{a}\right] = P\left[X > \frac{y-b}{a}\right] + P\left[X = \frac{y-b}{a}\right]$$
$$= 1 - F_X\left(\frac{y-b}{a}\right) + \cancel{P\left[X = \frac{y-b}{a}\right]} \to 0.$$

X

For continuous random variable, $\cancel{P}\left[X = \frac{y-b}{a}\right] = 0$. Hence,

$$\boxed{F_Y(y) = \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0, \\ 1 - F_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}}$$

step 2 :

Finally, fundamental theorem of calculus and chain rule gives

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right), & a > 0, \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}$$

Note that we can further simplify the final formula by using the $|\cdot|$ function:

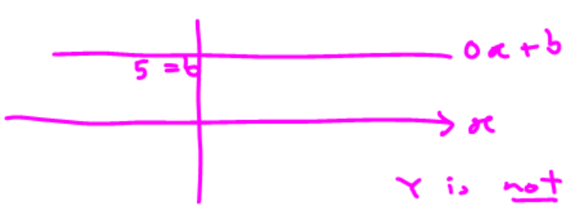
$y = ax + b$
$x = \frac{y-b}{a}$

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right), \quad a \ne 0.} \tag{24}$$

Graphically, to get the plots of $f_Y$, we compress $f_X$ horizontally by a factor of $a$, scale it vertically by a factor of $1/|a|$, and shift it to the right by $b$.

Of course, if $a = 0$, then we get the uninteresting degenerated random variable $Y \equiv b$.

$Y = aX + b$ where $a = 0$, $X$ is cont.




Y is not cont.

$$P[Y = y] = \begin{cases} 0, & y \ne b, \\ \underbrace{P[X \in (-\infty, \infty)]}_{1}, & y = b, \end{cases}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

**10.79.** Suppose $X \sim \mathcal{N}(m, \sigma^2)$ and $Y = aX + b$ for some constants $a$ and $b$. Then, we can use (24) to show that $Y \sim \mathcal{N}(am + b, a^2\sigma^2)$.

$$Y = aX + b \Rightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{\sqrt{2\pi}\,\sigma|a|} e^{-\frac{1}{2}\left(\frac{\frac{y-b}{a} - m}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\,\sigma|a|} e^{-\frac{1}{2}\left(\frac{y-(am+b)}{|a|\sigma}\right)^2}$$

$$Y \sim \mathcal{N}(am+b, \sigma^2 a^2) \qquad \sigma_Y = \sigma|a|$$

From earlier section(s),

$X \sim \mathcal{N}(m, \sigma^2)$

$\mathbb{E}X = m, \quad \operatorname{Var} X = \sigma^2.$

$Y = aX + b$

$\left.\begin{array}{l} \mathbb{E}Y = a\mathbb{E}X + b = am + b \\ \operatorname{Var} Y = a^2 \operatorname{Var} X = a^2\sigma^2 \end{array}\right\}$

$\sigma_Y = |a|\sigma$

Special Cases: ( [10.57] p. 144 )

(a) $Z \sim \mathcal{N}(0,1) \Rightarrow X = \underset{a=\sigma \quad b=m}{\sigma Z + m} \sim \mathcal{N}(m, \sigma^2)$

$a = 1/\sigma, \; b = -m/\sigma$

(b) $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Z = \frac{X-m}{\sigma} \sim \mathcal{N}(0,1)$

**Example 10.80.** Amplitude modulation in certain communication systems can be accomplished using various nonlinear devices such as a semiconductor diode. Suppose we model the nonlinear device by the function $Y = X^2$. If the input $X$ is a continuous random variable, find the density of the output $Y = X^2$.

Remark: The "square" func. is a very important func. in EC. For example, you need it for power calculation.

① $F_Y(y) \equiv P[Y \le y] = P[X^2 \le y]$

1.1 First note that $Y = X^2 \ge 0$. Therefore, for $y < 0$, $F_Y(y) = 0$.

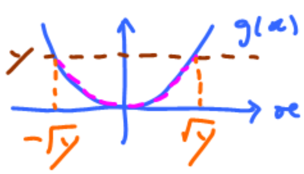
(When you are working on this type of question, always try to bring out this easy case first.)

Ex. $F_Y(-3) = P[Y \le -3] = P[X^2 \le -3] = 0.$

1.2 For any $y > 0$, $F_Y(y) = P[X^2 \le y] = P[\underset{a}{-\sqrt{y}} \le X \le \underset{b}{\sqrt{y}}] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

X is a cont. RV

Ex. $P[X^2 \le 4] = P[-2 \le X \le 2]$




1.3 For $y = 0$, $F_Y(0) = P[X^2 \le 0] = P[X = 0] = 0$

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & y > 0, \\ 0, & y \le 0. \end{cases} \xrightarrow[\frac{d}{dy}]{\text{step ②}} f_Y(y) = \begin{cases} \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}, & y > 0 \\ 0, & y \le 0 \end{cases}$$

add the case when $y = 0$.

(This is ok if we know $Y$ is a cont. RV)

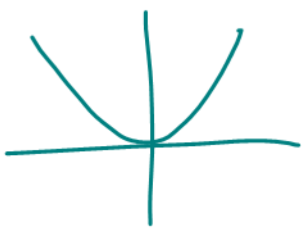
To check that a RV $Y$ is a cont. RV, there are two important techniques:

① Check that the cdf $F_Y(y)$ is a continuous function (for all $y$.)
(no jump)

② Check that $P[Y=y]=0$ (for any $y$).

Ex. Consider $Y = X^2$ when $X$ is a continuous RV

$$P[Y=y] = P[X^2=y] = \begin{cases} P[X=0] = 0, & y=0, \\ \underbrace{P[X=\sqrt{y}] + P[X=-\sqrt{y}]}_{0+0}, & y>0, \\ 0, & y<0, \end{cases}$$



There is no $X$ that satisfies $X^2$ = negative number.

Because $P[Y=y]=0$ for all $y$, we conclude that $Y$ is a cont. RV.

Ex. consider $Y = g(X)$ when $X$ is a cont. RV

Is $Y$ a continuous RV? and


Yes.

(The number of solutions for the eq. $y = g(x)$ is finite.)

Ex. $Y = \cos(X)$ when $X$ is a cont. RV

Is $Y$ a continuous RV?


Yes

(The number of solutions for the eq. $y=\cos(x)$ is countable

$$P[Y=0] = P[\cos(X)=0] = P[X = \tfrac{\pi}{2} + k\pi,\ k=0,\pm1,\pm2,\ldots]$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{P[X = \tfrac{\pi}{2} + k\pi]}_{0} = 0$$ )

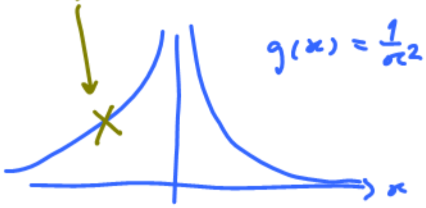
For each $y$ value, the collection of $x$ values that satisfy $y=g(x)$ is at most countable.

$\Rightarrow Y$ is a cont. RV.

In particular, two solutions when $y>0$ (one if we consider only $x>0$), no solution when $y\le 0$

Don't have to consider this side because $X\sim\mathcal{E}(\lambda)$ is always $\geq 0$.

**Example 10.81.** Suppose $X \sim \mathcal{E}(\lambda)$. Let $Y = \frac{1}{X^2}$. Find $f_Y(y)$.




$g(x) = \frac{1}{x^2}$

Step 1 $F_Y(y) \equiv P[Y \le y] = P\left[\frac{1}{X^2} \le y\right]$

1.1 Note that $\frac{1}{X^2} > 0 \Rightarrow F_Y(y) = 0$ for $y \le 0$

1.2 For $y>0$,

$F_Y(y) = P\left[\frac{1}{X^2} \le y\right] = P\left[X^2 \geq \frac{1}{y}\right] = P\left[X > \frac{1}{\sqrt{y}}\right] + \cancel{P\left[X < -\frac{1}{\sqrt{y}}\right]}$ $\; 0$ for $X\sim\mathcal{E}$

$= 1 - P\left[X \le \frac{1}{\sqrt{y}}\right] = 1 - F_X\left(\frac{1}{\sqrt{y}}\right)$

$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0, \\ 0, & \text{otherwise} \end{cases}$

$f_X\left(\frac{1}{\sqrt{y}}\right) = \begin{cases} \lambda e^{-\lambda/\sqrt{y}}, & \underbrace{\frac{1}{\sqrt{y}} > 0}_{\text{any } y>0 \text{ would satisfies this}}, \\ 0, & \text{otherwise} \end{cases}$

1.1 + 1.2 : $F_Y(y) = \begin{cases} 1 - F_X\left(\frac{1}{\sqrt{y}}\right), & y > 0, \\ 0, & \text{otherwise} \end{cases}$

**Exercise 10.82** (F2011)**.** Suppose $X$ is uniformly distributed on the interval $(1,2)$. ($X \sim \mathcal{U}(1,2)$.) Let $Y = \frac{1}{X^2}$.

(a) Find $f_Y(y)$.

(b) Find $\mathbb{E}Y$.

Step 2

$\underbrace{\frac{d}{dy}\frac{1}{\sqrt{y}}}= \frac{d}{dy} y^{-\frac{1}{2}} = -\frac{1}{2} y^{-\frac{3}{2}}$

$f_Y(y) = \begin{cases} +f_X\left(\frac{1}{\sqrt{y}}\right)\left(+\frac{1}{2}y^{-\frac{3}{2}}\right), & y>0, \\ 0, & \text{otherwise.} \end{cases}$

$= \begin{cases} \lambda e^{-\frac{\lambda}{\sqrt{y}}} \times \frac{1}{2} y^{-\frac{3}{2}}, & y>0, \\ 0, & \text{otherwise.} \end{cases}$

**Exercise 10.83** (F2011)**.** Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Suppose $Y = g(X)$, where $X \sim \mathcal{U}(-2,2)$.

Remark: The function $g$ operates like a **full-wave rectifier** in that if a positive input voltage $X$ is applied, the output is $Y = X$, while if a negative input voltage $X$ is applied, the output is $Y = -X$.

(a) Find $\mathbb{E}Y$.

(b) Plot the cdf of $Y$.

(c) Find the pdf of $Y$

| | Discrete | Continuous |
|---|---|---|
| Definition | $\equiv$ (minimal) support is countable<br>$\equiv \exists$ a countable set $S$ such that $P[X \in S] = 1$. (there exists)<br>$\equiv F_X(x)$ is a staircase func.  | $\equiv \underbrace{P[X=x]}_{p_X(x)} = 0 \quad \forall x$<br>$\equiv f_X(x)$ is real-valued (no $\delta$ function)<br>$\equiv F_X(x)$ is a continuous function  |
| $P[X \in B] =$ | $\sum\limits_{x \in B} p_X(x)$ | $\int\limits_B f_X(x)dx$ |
| $P[X = x] =$ | $p_X(x) = F(x) - F(x^-)$ | $0$ |
| Interval prob. | $P^X((a,b]) = F(b) - F(a)$<br>$P^X([a,b]) = F(b) - F(a^-)$<br>$P^X([a,b)) = F(b^-) - F(a^-)$<br>$P^X((a,b)) = F(b^-) - F(a)$ | $P[a < X \leq b] = P[a \leq X \leq b] = P[a \leq X < b] = P[a < X < b]$<br>$P^X((a,b]) = P^X([a,b])$<br>$= P^X([a,b)) = P^X((a,b))$<br>$= \int\limits_a^b f_X(x)dx = F(b) - F(a)$ |
| $\mathbb{E}X =$ | $\sum\limits_x x p_X(x)$ | $\int\limits_{-\infty}^{+\infty} x f_X(x)dx$ |
| For $Y = g(X)$, | $p_Y(y) = \sum\limits_{x:\, g(x)=y} p_X(x)$ | $f_Y(y) = \frac{d}{dy} P[g(X) \leq y]$.<br>Alternatively,<br>$f_Y(y) = \sum\limits_k \frac{f_X(x_k)}{\lvert g'(x_k)\rvert}$,<br>$x_k$ are the real-valued roots of the equation $y = g(x)$. |
| For $Y = g(X)$, $P[Y \in B] =$ | $\sum\limits_{x:g(x)\in B} p_X(x)$ | $\int\limits_{\{x:g(x)\in B\}} f_X(x)dx$ |
| $\mathbb{E}[g(X)] =$ | $\sum\limits_x g(x)p_X(x)$ | $\int\limits_{-\infty}^{+\infty} g(x)f_X(x)dx$ |
| $\mathbb{E}[X^2] =$ | $\sum\limits_x x^2 p_X(x)$ | $\int\limits_{-\infty}^{+\infty} x^2 f_X(x)dx$ |
| $\operatorname{Var} X =$ | $\sum\limits_x (x - \mathbb{E}X)^2 p_X(x)$ | $\int\limits_{-\infty}^{+\infty} (x - \mathbb{E}X)^2 f_X(x)dx$ |

Table 7: Important Formulas for Discrete and Continuous Random Variables